

## Voltage Ratio and Impedance Calculations for Coupled Inductors and Autotransformers

Ref. Electronic Engineering by Alley & Atwood p. 324  
Coupled Inductors

Jacques Audet Aug 2022  
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[https://en.wikipedia.org/wiki/Inductance#Mutual\\_inductance](https://en.wikipedia.org/wiki/Inductance#Mutual_inductance)

Portland University ECE221 Magnetically Coupled Circuits

### General Equations for Coupled Inductors

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \cdot \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

$V_x$  = voltages  
 $Z_{xx}$  = Impedances  
 $I_x$  = currents

$$V_1 = Z_{11} \cdot I_1 + Z_{12} \cdot I_2$$

$$V_2 = Z_{21} \cdot I_1 + Z_{22} \cdot I_2$$

### Calculation of $Z_{in}$ by setting $V_2 = 0$

$$0 = Z_{21} \cdot I_1 + Z_{22} \cdot I_2$$

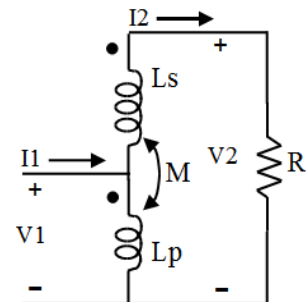
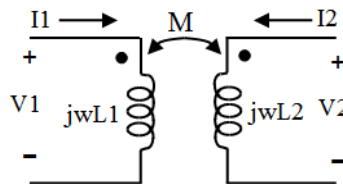
$$I_2 = -\frac{I_1 \cdot Z_{21}}{Z_{22}}$$

$$V_1 = Z_{11} \cdot I_1 + Z_{12} \cdot I_2$$

$$V_1 = \frac{I_1 \cdot (Z_{11} \cdot Z_{22} - Z_{12} \cdot Z_{21})}{Z_{22}}$$

$$Z_{in1} = \frac{Z_{11} \cdot Z_{22} - Z_{12} \cdot Z_{21}}{Z_{22}}$$

General equation for  $Z_{in}$  at port 1



For separate coupled inductors:

$$Z_{11} = j \cdot \omega \cdot L_1$$

$$Z_{12} = Z_{21} = j \cdot \omega \cdot M$$

$$Z_{22} = R + j \cdot \omega \cdot L_2$$

Where M is the mutual inductance for both cases

For the Autotransformer:

$$Z_{11} = j \cdot \omega \cdot L_1$$

$$Z_{12} = Z_{21} = -j \cdot \omega \cdot (L_1 + M)$$

$$Z_{22} = R + j \cdot \omega \cdot L$$

R = Load resistance at secondary

For both cases, with K = coupling coeff.

$$M = K \cdot \sqrt{L_1 \cdot L_2}$$

Notes:

The negative sign of  $Z_{12}$  and  $Z_{21}$  is used since  $i_2$  flows out of the polarity mark

$$L = L_1 + L_2 + 2 \cdot M \quad \text{Inductance at output}$$

### Exemple of Calculation of Zin1: R is on the secondary side (V2) for the autotransformer

$$L1 := 3 \quad L2 := 36 \cdot L1 \quad \underline{R} := 2450 \quad \underline{K} := 0.95 \quad M := K \cdot \sqrt{L1 \cdot L2} \quad \underline{L} := L1 + L2 + 2 \cdot M$$

Turns ratio = 6 so  $L2 / L1 = 36$

$f := 1, 1.1 \dots 100$  F in MHz and L, M in uH

$$\omega(f) := 2 \cdot \pi \cdot f$$

$$Z11(f) := j \cdot \omega(f) \cdot L1$$

$$Z12(f) := -j \cdot \omega(f) \cdot (L1 + M)$$

$$Z21(f) := Z12(f)$$

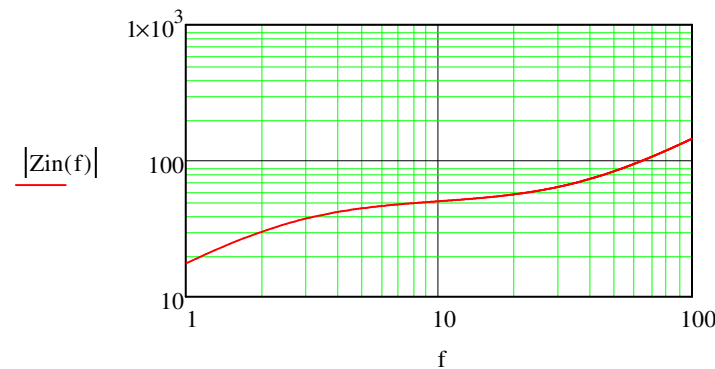
$$Z22(f) := R + j \cdot \omega(f) \cdot L$$

NOTE :

Interchanging L1 and L in Z11 and Z22 will calculate Zin2 (secondary side) with R on the primary side (V1)

$$Zin(f) := \frac{Z11(f) \cdot Z22(f) - Z12(f) \cdot Z21(f)}{Z22(f)}$$

General equation for Zin at port 1 (input)



$$|Zin(1)| = 17.671$$

$$|Zin(10)| = 50.639$$

$$|Zin(100)| = 145.717$$

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### Calculation of Av, the voltage transfer ratio

$$V1 = Z11 \cdot I1 + Z12 \cdot I2$$

$$V2 = Z21 \cdot I1 + Z22 \cdot I2$$

With  $V2=0$  Calculate  $Av = V2 / V1$   
R is on the sec. (V2)

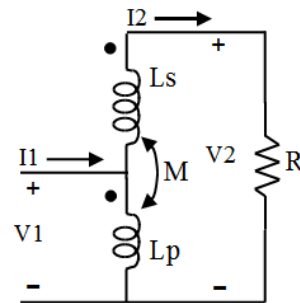
$$0 = Z21 \cdot I1 + Z22 \cdot I2$$

$$I1 = -\frac{I2 \cdot Z22}{Z21} \quad \text{Solve for } I1$$

$$V1 = Z11 \cdot I1 + Z12 \cdot I2$$

$$V1 = -\frac{I2 \cdot (Z11 \cdot Z22 - Z12 \cdot Z21)}{Z21}$$

Recall that:  $Av = \frac{I2}{V1} \cdot R$



$$Av = \frac{Z21 \cdot R}{Z12 \cdot Z21 - Z11 \cdot Z22}$$

General equation for  $Av = V2 / V1$ . This is NOT insertion loss !

### Example

$$\underline{L1} := 3 \quad \underline{L2} := 36 \cdot L1 \quad \underline{R} := 2450 \quad \underline{K} := 0.95 \quad \underline{M} := K \cdot \sqrt{L1 \cdot L2} \quad \underline{L} := L1 + L2 + 2 \cdot M$$

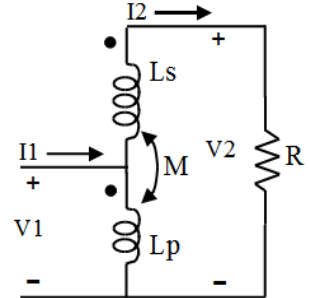
$$\underline{Z11}(f) := j \cdot \omega(f) \cdot L1$$

$$\underline{Z12}(f) := -j \cdot \omega(f) \cdot (L1 + M)$$

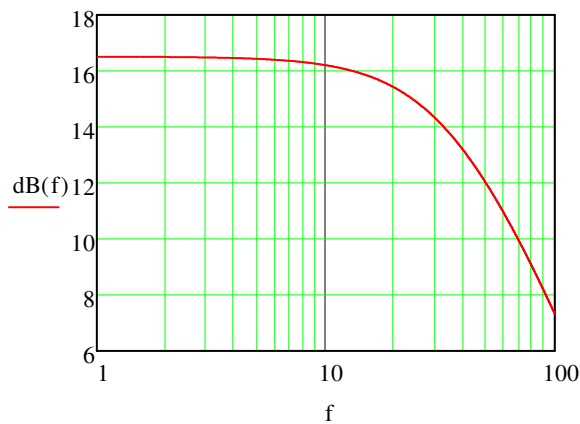
$$\underline{Z21}(f) := Z12(f)$$

$$\underline{Z22}(f) := R + j \cdot \omega(f) \cdot L$$

NOTE :  
Interchanging L1 and L in Z11 and Z22 will calculate V1 / V2 with R on the primary side (V1)



$$Av(f) := \frac{Z21(f) \cdot R}{Z12(f) \cdot Z21(f) - Z11(f) \cdot Z22(f)} \quad dB(f) := 20 \cdot \log(|Av(f)|)$$



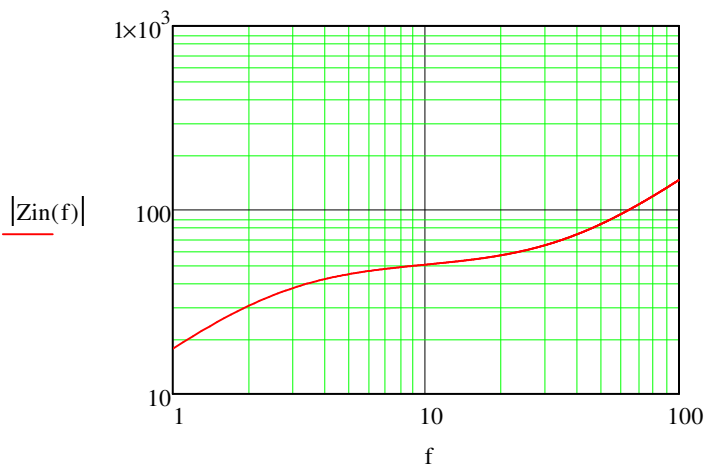
$$|Av(1)| = 6.698$$

$$|Av(10)| = 6.468$$

$$|Av(100)| = 2.327$$

### Zin and SWR Calculation at Input

$$\underline{Zin}(f) := \frac{Z11(f) \cdot Z22(f) - Z12(f) \cdot Z21(f)}{Z22(f)}$$



$$|Zin(1)| = 17.671$$

$$|Zin(10)| = 50.639$$

$$|Zin(100)| = 145.717$$

$$Zin(1) = 5.717 + 16.721i$$

$$Zin(10) = 43.791 + 25.43i$$

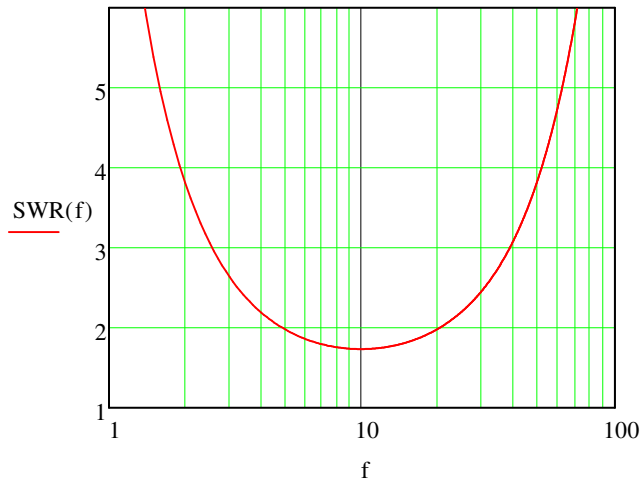
$$Zin(100) = 46.915 + 137.958i$$

## SWR Calculation based on Zin, at input

$Z_0 := 50$

$$SWR(f) = \frac{1 + \left| \frac{Z_{in}(f) - Z_0}{Z_{in}(f) + Z_0} \right|}{1 - \left| \frac{Z_{in}(f) - Z_0}{Z_{in}(f) + Z_0} \right|}$$

$$SWR(f) := \frac{2 \cdot |Z_0 + Z_{in}(f)|}{|Z_0 + Z_{in}(f)| - |Z_0 - Z_{in}(f)|} - 1$$



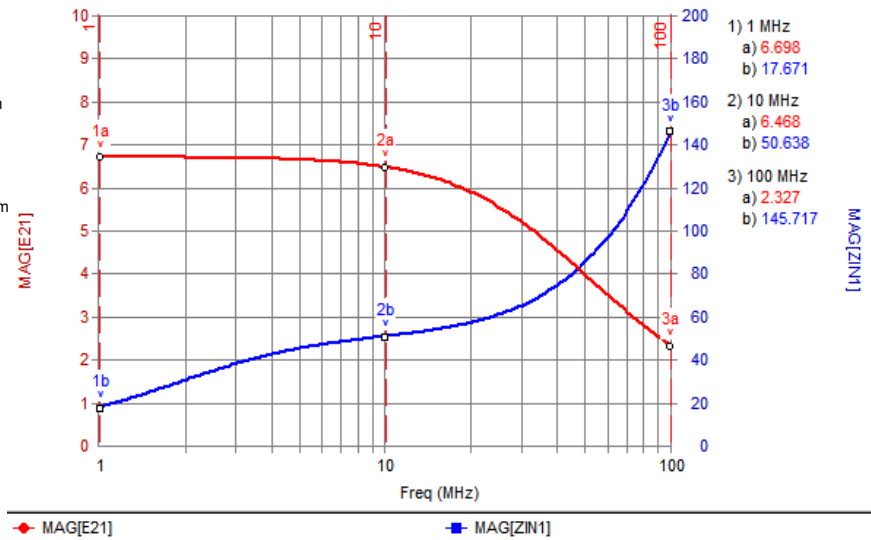
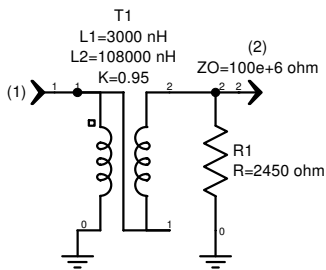
$$SWR(3.5) = 2.372$$

$$SWR(10) = 1.737$$

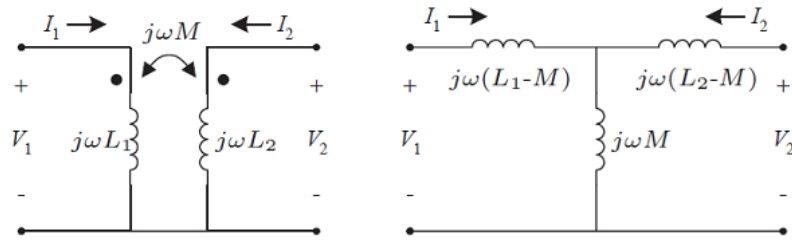
$$SWR(29) = 2.422$$

Ref: IndependantCoupledCoils-VS-Tapped coil-1.wsp

## Simulations



## Phasor Analysis: T-Equivalent



Frequency Domain (Phasors)

$$V_1 = j\omega L_1 I_1 + j\omega M I_2$$

$$V_2 = j\omega M I_1 + j\omega L_2 I_2$$

- The T-equivalent is only valid if bottom terminals are connected

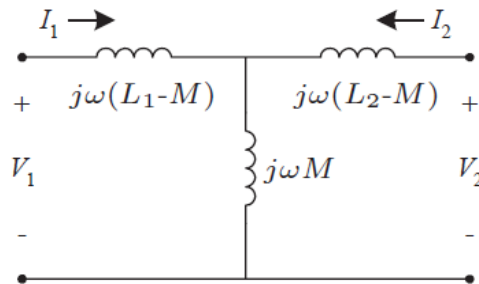
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### Equivalent Tee circuit for Independent Inductors

$$Z_{11} = j \cdot \omega \cdot L_1$$

$$Z_{12} = Z_{21} = j \cdot \omega \cdot M$$

$$Z_{22} = R + j \cdot \omega \cdot L_2$$



### Equivalent Tee Circuit for the Autotransformer

For the Autotransformer:

$$Z_{11} = j \cdot \omega \cdot L_1$$

$$Z_{12} = Z_{21} = -j \cdot \omega \cdot (L_1 + M)$$

$$Z_{22} = R + j \cdot \omega \cdot L$$

From comparisons with the independent inductor Tee circuit above

$$L_a = L_1 - (L_1 + M)$$

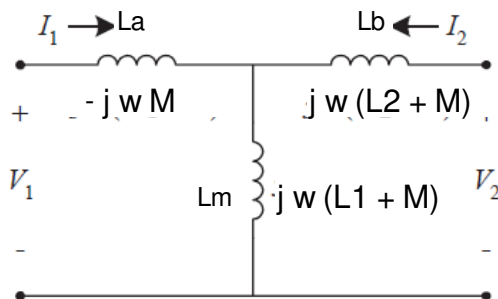
$$L_b = L - (L_1 + M) = L_2 + M$$

$$L_m = L_1 + M$$

$$M = K \cdot \sqrt{L_1 \cdot L_2}$$

$$L = L_1 + L_2 + 2 \cdot M$$

### Autotransformer equivalent Tee circuit



Example, using the same values as in page 3:

$$\underline{L1} := 3 \quad \underline{L2} := 36 \cdot L1 \quad \underline{R} := 2450 \quad \underline{K} := 0.95 \quad \underline{M} := K \cdot \sqrt{L1 \cdot L2} \quad \underline{L} := L1 + L2 + 2 \cdot M$$

M and L can be calculated:

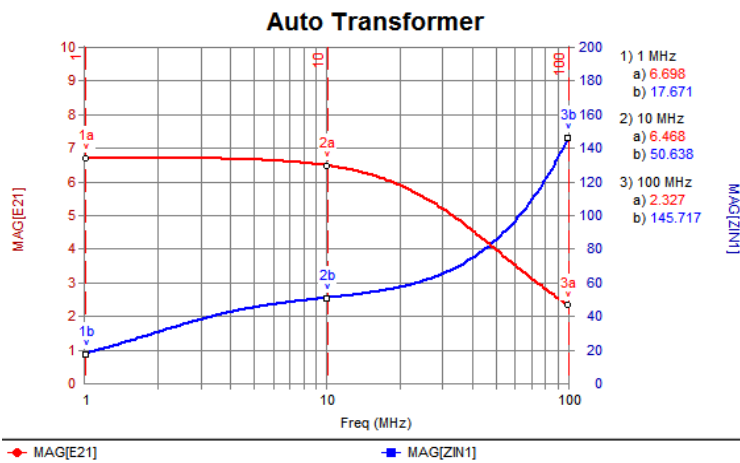
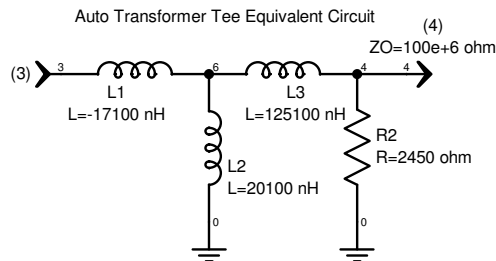
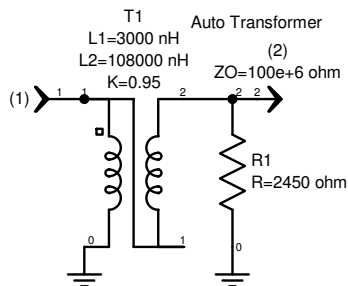
$$\underline{M} := K \cdot \sqrt{L1 \cdot L2} = 17.1 \quad \underline{L} := L1 + L2 + 2 \cdot M = 145.2$$

Tee Equivalent circuit:

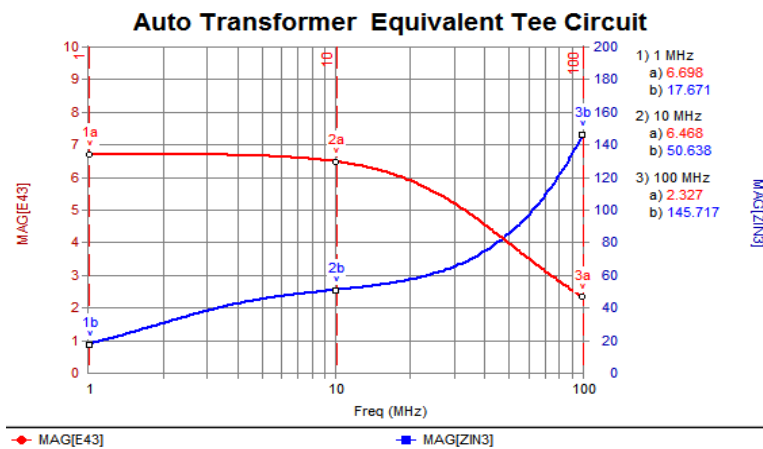
$$La := -M = -17.1$$

$$Lb := L2 + M = 125.1$$

$$Lm := L1 + M = 20.1$$



AutoTransformer Simulation



AutoTransformer Tee Equivalent Circuit Simulation

### Voltage Gain Variation vs Coupling Coefficient K

With  $K = 1$  (perfect coupling) the low frequency gain = 7.00

With  $K = 0.95$  the low frequency gain = 6.698 (dotted curves)

With  $K = 0.90$  the low frequency gain = 6.391 (solid curves)

